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HEAT CONDUCTION AND HEAT EXCHANGE IN TECHNOLOGICAL PROCESSES

ON THE INFLUENCE OF THE REINFORCEMENT STRUCTURE OF FIBROUS SHELLS OF REVOLUTION ON THE HEAT CONDUCTION IN THEM

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The initial boundary-value problem on the heat conduction in shells reinforced with fibers of constant cross section has been considered. It has been established that the specific, anisotropic, inhomogeneous properties of such a shell are determined by its heat conductivity dependent on the thermophysical properties of the phases of the composite material of the shell, the parameters of its reinforcement, and the geometry of this shell. The ways of reducing the three-dimensional problem on heat conduction to the two-dimensional one and the possibilities of reducing the dimension of this problem for thin shells of revolution reinforced symmetrically relative to their axis by two units have been determined. The stationary temperature fields of concrete thin shells of revolution with different Gaussian curvatures and different reinforcement structures have been compared. It is shown that the reinforcement structure and the geometry of a shell of revolution substantially influence the temperature distribution in this shell, which opens up a wide range of ways selecting designs of such shells with improved thermophysical parameters.

Elements accumulating and transferring heat are widely used in modern power plants, transport systems, jet engines of aircraft and spacecraft, laser facilities, and other apparatus. The potentialities of these elements, if they are made of homogeneous materials, have been practically exhausted. To substantially improve their properties, it is necessary to make them from composite materials with a discrete, continuous, or discrete-continuous distribution of the thermophysical parameters and heat sources. Polyreinforced multilayer structures, the materials of whose layers and the reinforcement phases of which have different thermophysical properties and the reinforcement trajectories in which are curvilinear (e.g., spiral reinforcement), possess the above-indicated properties. The obtaining of such layer-fiber structures presents no problems from the technological standpoint; these structures are widely used as effective load-bearing elements in transport systems and power plants and as elements of aircraft and spacecraft. However, the methods of investigating the heat conductivity of the indicated structures are only in their infancy. At present, only the simplest schemes of investigating cylindrically or spherically symmetric isotropic layered bodies [1–4] and unidirectional fibrous composites [5] have been developed, which prevents the search for structures that would be best in thermophysical parameters.

The general form of the linear heat-conduction equation for a fibrous shell in the curvilinear coordinate system x_i (i = 1, 2, 3) is as follows [6]:

$$CT_{,t} = (H_1H_2H_3)^{-1}\left[\sum_{i=1}^{3}\left(H_1H_2H_3H_i^{-1}\sum_{j=1}^{3}H_j^{-1}\Lambda_{ij}T_{,j}\right)_{,i}\right] + W\,, \tag{1}$$

where

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$$\Lambda_{ij} = \sum_{k=1}^{N} \omega_k \Omega^{-1} \left\{ l_{ki} l_{kj} \left[(\lambda_k - \lambda_m)\, \Omega + \lambda_m \right] + (-1)^{i+j}\, l_{kp} l_{kr} \left[(1 - \Omega)/\lambda_m + \Omega/\lambda_k \right]^{-1} \right\};$$

$$\Lambda_{33} = \left[(1 - \Omega)/\lambda_m + \sum_{k=1}^{N} \omega_k/\lambda_k \right]^{-1}, \quad p = 3 - i\,, \quad r = 3 - j\,, \quad i, j = 1, 2\,; \tag{2}$$

$$C = (1 - \Omega)\, c_m \rho_m + \sum_{k=1}^{N} \omega_k c_k \rho_k\,; \quad W = (1 - \Omega)\, w_m + \sum_{k=1}^{N} \omega_k w_k\,; \quad \Omega = \sum_{k=1}^{N} \omega_k\,;$$

$$l_{k1} = \cos \alpha_k\,, \quad l_{k2} = \sin \alpha_{k1}\,, \tag{3}$$

in this case, the following physical restrictions are obeyed:

$$\omega_k > 0 \quad (k = 1, 2, ..., N)\,, \quad \Omega < 1\,. \tag{4}$$

The most popular modern technologies of obtaining products from reinforced composites involve the use of fibers having a constant cross section along their length; therefore, the reinforcement parameters ω_k and α_k cannot be independent. The condition of constancy of the cross sections of the fibers of the kth family (kth fibers) in the curvilinear orthogonal coordinate system x_i has the form [7]

$$(H_2 H_3 \omega_k \cos \alpha_k)_{,1} + (H_1 H_3 \omega_k \sin \alpha_k)_{,2} = 0\,, \quad k = 1, 2, ..., N\,. \tag{5}$$

Consequently, the total system of linear heat-conduction equations for a reinforced shell or a plate includes relations (1)–(5) that should be supplemented with initial and boundary temperature conditions [2–4], and, at the edge S_k, where the kth fibers enter the shell, the value of the function ω_k should be prescribed [7].

The heat-conductivity coefficients and heat capacity of a composite material as well as the power density of its internal heat sources are usually assumed to be known from experiments. However, it is seen from relations (2) and (3) that the effective heat-conductivity coefficients Λ_{ij} (i, j = 1, 2, 3), the reduced heat capacity C, and the reduced power density of the internal heat sources W of a composite fibrous material substantially depend not only on the thermophysical parameters of its phases, but also on the structural parameters of a reinforcement — its direction α_k and density ω_k. While the influence of the thermophysical parameters of the phases of simple composites with a rectilinear reinforcement and the influence of the reinforcement density on the heat conductivity of these composites was investigated experimentally and theoretically [5], the influence of a complex reinforcement structure of composites having a complex geometry on their heat conduction was not investigated at all. (The material of shells with a complex reinforcement structure, which are usually obtained by applying a winding or by facing, possesses anisotropic and specific inhomogeneous properties depending on the method of obtaining such a reinforced shell; this is embodied in relations (1)–(5).) Qualitative and quantitative analyses of the indicated influence allow one to solve a number of problems arising in the process of designing products with the use of fibrous composites: to determine the region of significant influence of the reinforcement structure of a composite on its temperature field; select optimum and rational, in thermophysical parameters, reinforcement structures; develop criteria for such structures; and so on.

Heat Conduction in Thin Shells of Revolution Reinforced Symmetrically to Their Axis. To the initial boundary-value problem corresponds the heat-conduction equation (1) that can be integrated by different approximate methods, e.g., the method of straight lines [8]. However, Eq. (1) is three-dimensional and the problem on the heat conductivity of shells and plates is two-dimensional. The three-dimensional problem on heat conduction can be reduced to the two-dimensional one by the Bubnov–Galerkin method. Actually, let the temperature at the "outer" face of a shell ($x_3 = x_3^0 > 0$) be equal to $T_+(t, x_1, x_2)$ and the temperature of the "inner" surface ($x_3 = -x_3^0 < 0$) be equal to $T_-(t, x_1, x_2)$; then, according to the Bubnov–Galerkin method, the temperature of the whole shell will be equal to

$$T(t, x_1, x_2, x_3) = \left(2x_3^0\right)^{-1} \left[\left(x_3 + x_3^0\right)T_+ - \left(x_3 - x_3^0\right)T_-\right] + \sum_{n=0}^{\infty} \left\{T_{cn}(t, x_1, x_2) \times\right.$$

$$\left. \times \cos\left[(2n+1)\,\pi x_3/(2x_3^0)\right] + T_{sn}(t, x_1, x_2) \sin\left[2n\pi x_3/(2x_3^0)\right]\right\}, \tag{6}$$

and Eq. (1) will take the operator form

$$L(T) = 0\,. \tag{7}$$

Let us substitute representation (6) into Eq. (7) and lay down the condition that

$$\int_{-x_3^0}^{x_3^0} L(T) \cos\frac{(2n+1)\,\pi x_3}{2x_3^0}\, dx_3 = 0\,, \quad \int_{-x_3^0}^{x_3^0} L(T) \sin\frac{n\pi x_3}{x_3^0}\, dx_3 = 0\,, \quad n = 0, 1, 2, \ldots\,. \tag{8}$$

Integration of (8) gives the system of differential equations for the functions T_{cn} and T_{sn} that are dependent only on the two spatial variables x_1 and x_2 and the time t. If the heat exchange between the shell and the environment is realized through the faces of the shell by the convective Newton law

$$\overset{+}{q}_3 = \mu_+ (T_+ - T_{+\infty})\,, \quad -\overset{-}{q}_3 = \mu_- (T_- - T_{-\infty})\,, \tag{9}$$

where $\overset{\pm}{q}_3 = q_3(t,\ x_1,\ x_2,\ \pm x_3^0)$ and $T_\pm = T\ (t,\ x_1,\ x_2,\ \pm x_3^0)$, Eq. (1) is solved in the following form:

$$T = \sum_{n=0}^{\infty} T_n(t, x_1, x_2)\,(x_3)^n = T_0(t, x_1, x_2) + T_1(t, x_1, x_2)\,x_3 + T_2(t, x_1, x_2)\,(x_3)^2 + \ldots\,. \tag{10}$$

Let us substitute expression (10) into (7) and lay down the condition that

$$\int_{-x_3^0}^{x_3^0} L(T)\,(x_3)^n\, dx_3 = 0\,, \quad n = 0, 1, 2, \ldots\,. \tag{11}$$

This system should be supplemented with two boundary conditions (9) that, with allowance for the Fourier law, will take the form

$$-(\pm \overset{\pm}{\Lambda}_{33}) \sum_{n=1}^{\infty} T_n(t, x_1, x_2)\,(\pm x_3^0)^{n-1} = \mu_\pm \sum_{n=0}^{\infty} T_n(t, x_1, x_2)\,(\pm x_3^0)^n - \mu_\pm T_{\pm\infty}\,, \tag{12}$$

where $\overset{\pm}{\Lambda}_{33} = \Lambda_{33}(x_1,\ x_2,\ \pm x_3^0)$. The system of equations (11)–(12) allows one to determine the functions T_n that are dependent on only the time and two spatial variables. This system should be supplemented with initial and boundary conditions (conditions at the edges of the shell) represented in the form of (7) and (11).

Equations (11) and (12) for thin shells, which are used frequently as elements of power plants and other apparatus, are the most simple because, in this case, the Lamé parameters H_i (i = 1, 2, 3), the power densities of the internal heat sources in the phases ω_m and ω_k of a composite, the reinforcement parameters ω_k and α_k (k = 1, 2, ..., N), and, consequently, the functions C, W, and λ_{ij} can be considered as independent of x_3 [7] and expansion (10) can be reduced to three terms [9].

We will consider a thin shell of constant thickness $H = 2h$ ($h = x_3^0$) with reinforced equidistant surfaces. Let x_3 be the distance from the middle surface of the shell ($x_3 = 0$) to a reinforced layer; in this case, $H_3 = 1$ m. It will be assumed that H_1, H_2, and the reinforcement parameters are independent of x_3 and expansion (10) includes only three terms. Then, conditions (12) and Eq. (11), at $n = 0$, will take the form [6]

$$\mp \Lambda_{33} (T_1 \pm 2hT_2) = \mu_{\pm} (T_0 \pm hT_1 + h^2 T_2 - T_{\pm\infty}) , \tag{13}$$

$$C\Theta_{,t} = (H_1 H_2)^{-1} \sum_{i=1,2} \left[H_1 H_2 H_i^{-1} \sum_{j=1,2} (\Lambda_{ij} H_j^{-1} \Theta_{,j}) \right]_{,i} - \mu_- (T_0 - hT_1 + h^2 T_2 -$$

$$- T_{-\infty}) - \mu_+ (T_0 + hT_1 + h^2 T_2 - T_{+\infty}) + HW(t, x_1, x_2) , \tag{14}$$

where $\Theta = \int_{-h}^{h} T dx_3 = H(T_0 + h^2 T_2/3)$. From equalities (13) we obtain, having expressed the functions T_1 and T_2 in terms of Θ and $T_{\pm\infty}$ and having eliminated them from (14), a heat-conduction equation containing only one unknown function Θ:

$$C\Theta_{,t} = \left(H_1 H_2\right)^{-1} \sum_{i=1,2} \left[H_1 H_2 H_i^{-1} \sum_{j=1,2} \left(\Lambda_{ij} H_j^{-1} \Theta_{,j} \right) \right]_{,i} + A\Theta + B_- T_{-\infty} + B_+ T_{+\infty} + HW(t, x_1, x_2) , \tag{15}$$

$$T_0 = \Theta/H - h^2 T_1/3 , \quad T_1 = [(a_{22} - a_{12})\, \Theta/H - a_{22} T_{+\infty} + a_{12} T_{-\infty}]/\Delta ,$$

$$T_2 = [(a_{11} - a_{21})\, \Theta/H + a_{21} T_{+\infty} - a_{11} T_{-\infty}]/\Delta ,$$

where

$$A = -(\mu_+ + \mu_-)\, [1/H + h\, (a_{11} - a_{21})/(3\Delta)] + (\mu_- - \mu_+)\, (a_{22} - a_{12})/(2\Delta) ;$$

$$B_+ = \mu_+ + (\mu_+ - \mu_-)\, h a_{22}/\Delta - (\mu_+ + \mu_-)\, Hh a_{21}/(3\Delta) ;$$

$$B_- = \mu_- + (\mu_- - \mu_+)\, h a_{12}/\Delta + (\mu_+ + \mu_-)\, Hh a_{11}/(3\Delta) ;$$

$$\Delta = a_{11} a_{22} - a_{12} a_{21} ; \quad a_{11} = -(m_+ + h) ; \quad a_{12} = -H(m_+ + h/3) ;$$

$$a_{21} = m_- + h ; \quad a_{22} = -H(m_- + h/3) ; \quad m_{\pm} = \Lambda_{33}/\mu_{\pm} .$$

We may proceed analogously in the case where other boundary conditions are set on the faces of the shell: conditions for the temperature and the heat flow or mixed conditions. For example, when heat-flow values are prescribed on these surfaces ($\mp\Lambda_{33}(T_1 \pm 2hT_2) = q_3^{\pm}$), the heat-conduction equation will take the simpler form [6]

$$C\Theta_{,t} = \left(H_1 H_2\right)^{-1} \sum_{i=1,2} \left[H_1 H_2 H_i^{-1} \sum_{j=1,2} \left(\Lambda_{ij} H_j^{-1} \Theta_{,j} \right) \right]_{,i} - \left(q_3^+ - q_3^- \right) + HW , \tag{16}$$

$$T_0 = \Theta/H - h^2 T_1/3 , \quad T_1 = \left(q_3^- - q_3^+ \right) \big/ \left(2\Lambda_{33} \right), \quad T_2 = -\left(q_3^- + q_3^+ \right) \big/ \left(2H\Lambda_{33} \right).$$

Consequently, for thin shells, the three-dimensional heat-conduction equation (1) is reduced with a sufficient accuracy to a two-dimensional equation of the type of (15) and (16) for the one unknown function Θ dependent on only the time t and the two spatial variables x_1 and x_2.

Evidently, to formulate the initial boundary-value problem corresponding to Eqs. (15) and (16), it is necessary to integrate the initial and boundary heat conditions (set at the edges of the shell) over the thickness of the shell, whereupon we will obtain the initial and boundary conditions for the function Θ.

The influence of the reinforcement structure on the temperature field can be most clearly demonstrated by the example of simple composites; therefore, we will further consider only thin shells of revolution with an axially symmetric reinforcement of equidistant surfaces. The dimension of the problem for such composites can be decreased by one more unit. We will consider a shell in the rectangular coordinate system y_1, y_2, y_3, in which its rotation axis will be coincident with the Oy_1 axis; in this case, the parameter x_1 will determine the distance from a point of the middle surface to the plane $y_1 = 0$ ($x_1^0 \le x_1 \le x_1^1$) and x_2 will be a polar angle in the cylindrical coordinate system related to the rotation axis of the shell ($0 \le x_2 < 2\pi$). Then the Lamé parameters will take the form

$$H_1 = \sqrt{1 + (R'(x_1))^2}\,, \quad H_2 = R(x_1)\,, \tag{17}$$

and all the coefficients in equations of the type (15) and (16) will be dependent on only the variable x_1.

Since the initial boundary-value problem corresponding to Eqs. (15) and (16) is linear and its solution for the variable x_2 is periodic, we will expand Θ and the known functions $T_{\pm\infty}$, $\overset{\pm}{q}_3$, and W in a Fourier series about x_2 [10]:

$$\Theta(t, x_1, x_2) = \Theta_{00}(t, x_1) + \sum_{n=1}^{\infty} \Big[\Theta_{1n}(t, x_1)\cos(nx_2) + \Theta_{2n}(t, x_1)\sin(nx_2)\Big],$$

$$T_{\pm\infty}(t, x_1, x_2) = T_{00}^{(\pm)}(t, x_1) + \sum_{n=1}^{\infty} \Big[T_{1n}^{(\pm)}(t, x_1)\cos(nx_2) + T_{2n}^{(\pm)}(t, x_1)\sin(nx_2)\Big],$$

$$\overset{\pm}{q}_3(t, x_1, x_2) = q_{00}^{(\pm)}(t, x_1) + \sum_{n=1}^{\infty} \Big[q_{1n}^{(\pm)}(t, x_1)\cos(nx_2) + q_{2n}^{(\pm)}(t, x_1)\sin(nx_2)\Big],$$

$$w_{\mathrm{m}}(t, x_1, x_2) = \overset{\mathrm{m}}{w}_{00}(t, x_1) + \sum_{n=1}^{\infty} \Big[\overset{\mathrm{m}}{w}_{1n}(t, x_1)\cos(nx_2) + \overset{\mathrm{m}}{w}_{2n}(t, x_1)\sin(nx_2)\Big], \tag{18}$$

$$w_k(t, x_1, x_2) = w_{00}^{(k)}(t, x_1) + \sum_{n=1}^{\infty} \Big[w_{1n}^{(k)}(t, x_1)\cos(nx_2) + w_{2n}^{(k)}(t, x_1)\sin(nx_2)\Big],$$

$$k = 1, 2, \ldots, N\,.$$

Let us substitute these expansions, e.g., into Eq. (15) and select terms with cofactors $\cos(nx_2)$ and $\sin(nx_2)$. Then, for each $n \ge 0$, we will obtain the following systems of equations determining the functions Θ_{00}, Θ_{1n}, and Θ_{2n}:

$$C\Theta_{00,t} = H_1^{-2}\Lambda_{11}\Theta_{00,11} + (H_1 R)^{-1}(R\Lambda_{11}/H_1)'\,\Theta_{00,1} +$$

$$+ A\Theta_{00} + B_+ T_{00}^{(+)} + B_- T_{00}^{(-)} + H\left[(1 - \Omega)\,\overset{\mathrm{m}}{w}_{00} + \sum_{k=1}^{N} \omega_k w_{00}^{(k)}\right], \tag{19}$$

$$C\Theta_{in,t} = H_1^{-2}\Lambda_{11}\Theta_{in,11} + (H_1R)^{-1}\,(R\Lambda_{11}/H_1)'\,\Theta_{in,1} - (-1)^i\,2\,(H_1R)^{-1}\,n\Lambda_{12}\times$$

$$\times\,\Theta_{jn,1} + (A - R^{-2}\Lambda_{22}n^2)\,\Theta_{in} - (-1)^i\,(H_1R)^{-1}\,n\Lambda_{12}'\Theta_{jn} + B_+T_{in}^{(+)} + B_-T_{in}^{(-)} +$$

$$+\,H\left[(1-\Omega)\,w_{in}^{\mathrm{m}} + \sum_{k=1}^{N}\omega_k w_{in}^{(k)}\right],\quad j = 3 - i\,,\quad i = 1, 2\,,\quad n = 1, 2, 3, \ldots\,. \tag{20}$$

Having expanded, by analogy with (18), the initial and boundary conditions for the function Θ, we will obtain the corresponding initial and boundary conditions for the functions Θ_{00}, Θ_{1n}, and Θ_{2n}, which will be dependent on only one of the variables x_1 or t respectively. Evidently, the initial boundary-value problem corresponding to Eqs. (19) and (20) is analogous to the nonstationary one-dimensional problem on heat conduction, for integration of which there are well-developed methods [11].

If a heat action is axially symmetric, $\Theta = \Theta_{00}\ (t,\ x_1)$ and the solution of the initial boundary-value problem on the heat conductivity of a shell is reduced to the integration of Eq. (19) at corresponding initial and boundary conditions, i.e., the problem is reduced to the nonstationary, one-dimensional, heat-conduction problem. If the faces of the shell are heat-insulated ($q_3^{\pm} = 0$), it follows from (16) that $T_1 = T_2 = 0$ and $T_0 = \Theta/H$, i.e., it may be assumed with an accuracy to the third order of magnitude that the temperature is constant throughout the thickness of the shell; in this case, the coefficients in expansion (18) of the function Θ will be determined from system (19)–(20) at $A = B_+ = B_- = 0$. In the case where a heat action is axially symmetric and the faces of the shell are heat-isolated, the heat-conduction equation is reduced to the form

$$CT_{,t} = (H_1R)^{-1}\,(RH_1^{-1}\Lambda_{11}T_{,1})_{,1} + (1-\Omega)\,w_{\mathrm{m}} + \sum_{k=1}^{N}\omega_k w_k\,, \tag{21}$$

which follows from (19) and the relation $T = T_0 = \Theta/H$. In Eq. (21), all the coefficients are dependent on only x_1 and the functions T, ω_{m}, and ω_k are dependent on t and x_1.

Thus, for thin shells, the dimension of the heat-conduction equation (1) can be decreased by unity with a sufficient degree of accuracy by eliminating from consideration the spatial variable x_3 [9] determining the distance from the middle surface of the shell to a reinforced layer. For shells of revolution with an axially symmetric reinforcement, the dimensionality of Eqs. (1) can be decreased by two units by eliminating from consideration not only the variable x_3, but also the variable x_2 determining the positions of points on the middle surface in the peripheral direction.

Analysis of Some Solutions. To demonstrate the influence of the reinforcement structure of a shell of revolution on the temperature distribution in it more dramatically, we will consider simple examples, namely, the stationary axially symmetric problem on the heat conduction in thin shells of revolution of different Gaussian curvature K with surfaces reinforced with fibers of different orientation.

Let us consider the heat conduction in shells of revolution of the three most characteristic types: first type — conic shells with a zero Gaussian curvature ($K = 0$), for which

$$R\,(x_1) = \left[\left(x_1 - x_1^0\right)R_1 - \left(x_1 - x_1^1\right)R_0\right]\Big/\left(x_1^1 - x_1^0\right); \tag{22}$$

second type — shells of the type of an elliptic paraboloid of revolution with a positive Gaussian curvature ($K > 0$), for which

$$R\,(x_1) = a\sqrt{x_1 - c} + b\,, \tag{23}$$

where

$$a = (R_1 - R_0)\left(\sqrt{x_1^1 - c} - \sqrt{x_1^0 - c}\right)^{-1};\quad b = R_0 - a\sqrt{x_1^0 - c}\,;\quad c < x_1^0\,; \tag{24}$$

third type — shells of the type of a one-sheeted hyperboloid of revolution with a negative Gaussian curvature ($K < 0$), for which

$$R(x_1) = \sqrt{a^2 + b^2 (x_1 - c)^2}\,, \tag{25}$$

where

$$a^2 = \left[R_0^2\left(x_1^1 - c\right)^2 - R_1^2\left(x_1^0 - c\right)^2\right] \Big/ \left[\left(x_1^1 - c\right)^2 - \left(x_1^0 - c\right)^2\right];$$

$$b^2 = \left(R_1^2 - R_0^2\right) \Big/ \left[\left(x_1^1 - c\right)^2 - \left(x_1^0 - c\right)^2\right]; \quad c^0 \le c \le c^1\,; \tag{26}$$

$$c^0 = \left(R_1 x_1^0 - R_0 x_1^1\right) \Big/ \left(R_1 - R_0\right); \quad c^1 = \left(R_0 x_1^1 + R_1 x_1^0\right) \Big/ \left(R_1 + R_0\right).$$

It is assumed that, in relations (22), (24), and (26), $x_1^1 > x_1^0$ and $R_1 > R_0$; $R_i = R(x_1^i)$, where $i = 0, 1$; the parameter c determines the families of shells (23), (25); in this case, at $c \to -\infty$, an elliptic-paraboloid shell and, at $c \to c^0$ or c^1, a one-sheeted-hyperboloid shell degenerate into conic shells.

The stationary, axially symmetric heat conduction in a shell with an axially symmetric reinforcement and heat-isolated faces, in which heat sources are absent, has the form (see (21))

$$\left(H_1 R\right)^{-1} \left(R H_1^{-1} \Lambda_{11} T'\right)' = 0\,. \tag{27}$$

Integration of this equation gives

$$R H_1^{-1} \Lambda_{11} T' = -R q_1 \equiv -q_* = \text{const}\,. \tag{28}$$

If a shell is reinforced with fibers of constant cross section symmetrically relative to its axis, the condition of constancy of reinforcement cross sections (5) can be integrated:

$$R \omega_k \cos \alpha_k = \omega_{*k} = \text{const}\,, \quad k = 1, 2, \ldots, N\,, \tag{29}$$

where the parameter ω_{*k} determining, with an accuracy to any constant factor, the total area of the cross sections of the kth fibers [7] is an initial parameter of the problem. (From all the axially symmetric reinforcements of shells of revolution, the degenerate case of peripheral packing of fibers ($\alpha_k = \pi/2$) should be set apart; in this case, equality (29) holds identically at $\omega_{*k} = 0$ and the function ω_k can be arbitrary if the physical restrictions (4) are obeyed).

If the reinforcement directions ω_k are known, Eq. (27) and the boundary conditions $T(x_1^0)$ and $T(x_1^1)$ form, in combination with expressions (2) and (29), a linear two-point boundary problem and Eq. (28) and the boundary conditions $T(x_1^i)$ and $q_1(x_1^i)$ ($i = 0$ or $i = 1$) form a linear Cauchy problem on the temperature T, for the solution of which there are well-developed numerical methods [8, 11].

Let us consider the influence of the reinforcement structure of concrete shells of revolution having identical characteristic sizes (the length along the rotation axis and the radii of two edges) on their temperature field at identical boundary conditions. For comparison of reinforcement structures, we will use as a criterion the total consumption of kth reinforcements, determined as

$$\Omega_k = \int_V \omega_k dV = 2\pi H \int_{x_1^0}^{x_1^1} \omega_k(x_1)\, R(x_1)\, H_1(x_1)\, dx_1\,. \tag{30}$$

We investigated composite metal shells bounded by edges of radius R_0 and R_1: $R_1 = 5R_0$ ($x_1^0 = 0$ and $x_1^1 = 10R_0$), made of copper ($\lambda_m = 400$ W/(m·K), $c_m = 419$ J/(kg·K), $\rho_m = 8940$ kg/m^3 [12]), which were reinforced with steel fibers of two (N = 2) families (λ_k = 45 W/(m·K), c_k = 568 J/(kg·K), ρ_k = 7780 kg/m^3, k = 1, 2). It will be assumed that, at the edge x_1^0, the temperature $T(x_1^0)$ = 300°C and the heat flow $R_0 q_1(x_1^0)$ = 5000 W/m.

We now analyze the effect of the following most characteristic reinforcement structures: first-type — meridional reinforcement: fibers of both families are packed with equal densities along the meridional directions ($\alpha_1 = \alpha_2 = 0$); second and third types — spiral reinforcements: fibers are packed along the symmetric meridional directions ($\alpha_1 = -\alpha_2$) at angles $\alpha_1 = \pi/6$ and $\alpha_1 = \pi/3$ respectively; fourth, fifth, and sixth types — peripheral reinforcements: fibers are packed along the peripheral directions ($\alpha_1 = -\alpha_2 = \pi/2$) but with different reinforcement-density distributions ω_k; seventh type — meridional-peripheral reinforcement: the first-family fibers are packed along the meridional direction and the second-family fibers are packed with a constant density along the peripheral direction (ω_2 = const). In addition, we will consider one more (eighth) type of reinforcement, which is most natural for one-sheeted hyperboloid shells: fibers are packed along the asymptotic directions [13]

$$\operatorname{tg}\alpha_k = (-1)^k \sqrt{RR''/(1+R'^2)}\,, \quad k = 1, 2 \tag{31}$$

(in this case, the reinforcement trajectories are rectilinear).

Since the function ω_k can be arbitrary in the case of peripheral packing of reinforcements [7], the densities of the fourth-, fifth-, and sixth-type reinforcements will be determined by the linear law

$$\omega_k(x_1) = \left[\left(x_1 - x_1^0\right)\omega_k^1 - \left(x_1 - x_1^1\right)\omega_k^0\right] \Big/ \left(x_1^1 - x_1^0\right), \quad \omega_1(x_1) = \omega_2(x_1)\,, \tag{32}$$

(here, $\omega_k^i = \omega_k(x_1^i)$, where i = 0, 1); in this case, it will be assumed that $\omega_k^0 = \omega_k^1$, i.e., ω_k = const for the fourth-type reinforcement, $\omega_k^0 \neq 0$ and $\omega_k^1 = 0$ for the fifth-type reinforcement, and $\omega_k^0 = 0$ and $\omega_k^1 \neq 0$ (k = 1, 2) for the sixth-type reinforcement. The values of ω_{*k} in (29) and ω_k^i in (32) will be selected such that the total number of fibers used (30) will be equal in all the reinforcement structures considered and, additionally, the maximum total reinforcement density ($\Omega = \omega_1 + \omega_2$) will be not larger than 0.8 (max $\Omega \leq 0.8$).

Figure 1 shows the temperature distribution in a conic shell, an elliptic-paraboloid shell ($c = -0.01R_0$), and a one-sheeted-hyperboloid shell ($c = 1.5R_0$) with different reinforcement structures. The numbers of curves 1–7 correspond to the ordinal numbers of the above-described types of reinforcement structures; curve 8 in Fig. 1b corresponds to the meridional reinforcement structure of an elliptic-paraboloid shell with $c = -10R_0$ (see (23)); curves 8, 9, and 10 in Fig. 1c characterize the temperature distribution in one-sheeted-hyperboloid shells of different geometries (c/R_0 = 1.5, 0, and −2) with a reinforcement structure along the asymptotic directions (31).

The comparison of curves 1–3 presented in Fig. 1 allows one to determine the dependence of the temperature distribution on the direction of reinforcement at one and the same reinforcement density. (Actually, it follows from (29) that, in all structures reinforced at constant angles different from $\pi/2$, the reinforcement density is equal to $\omega_k = R_0\omega_k^0/R$.) For example, the temperature difference at the edge of shells with a meridional (curve 1) and a spiral ($\alpha_k = \pm\pi/3$, curve 3) reinforcement structure ($T^{(1)}(x_1^1) - T^{(3)}(x_1^1)$) (hereinafter, the index in parentheses denotes the number of a curve) comprises 26.5, 15.3, and 109.5°C in a conic shell, an elliptic-paraboloid shell, and a one-sheeted-hyperboloid shell respectively.

The comparison of curves 4–6 in Fig. 1 allows one to determine the dependence of the temperature on the reinforcement density ω_k at a constant (peripheral) direction of reinforcement. In this case, the difference $T^{(6)}(x_1^1) - T^{(5)}(x_1^1)$ comprises 21.5, 8.7, and 133.9°C respectively in the cases illustrated in Fig. 1a, b, and c.

The comparison of curves 1–6 with curve 7 in Fig. 1 and curve 8 in Fig. 1c shows that the change in both the direction and density of reinforcement significantly influence the temperature-field distribution in a shell.

The comparison of identical curves 1–7 in Fig. 1 shows that not only the type of reinforcement but also the geometry of a shell significantly influence the temperature field of this shell. For example, with change from the meridional reinforcement to the fifth-type reinforcement, the temperature difference at the edges ($T(x_1^0) - T(x_1^1)$) of a conic shell (Fig. 1a) changes from 52°C to 101°C, i.e., the temperature difference at the shell edge x_1^1 comprises 49°C ($T^{(1)}(x_1^1) - T^{(5)}(x_1^1)$ = 49°C); in an elliptic-paraboloid shell ($c = -0.01R_0$, Fig. 1b), the temperature difference at the edges increases from 38.5 to 64.5°C and the temperature difference at the edge x_1^1 comprises only 26°C; and in a one-sheeted-hyperboloid shell ($c = 1.5R_0$, Fig. 1c), the temperature difference at the edges changes from 139.9°C

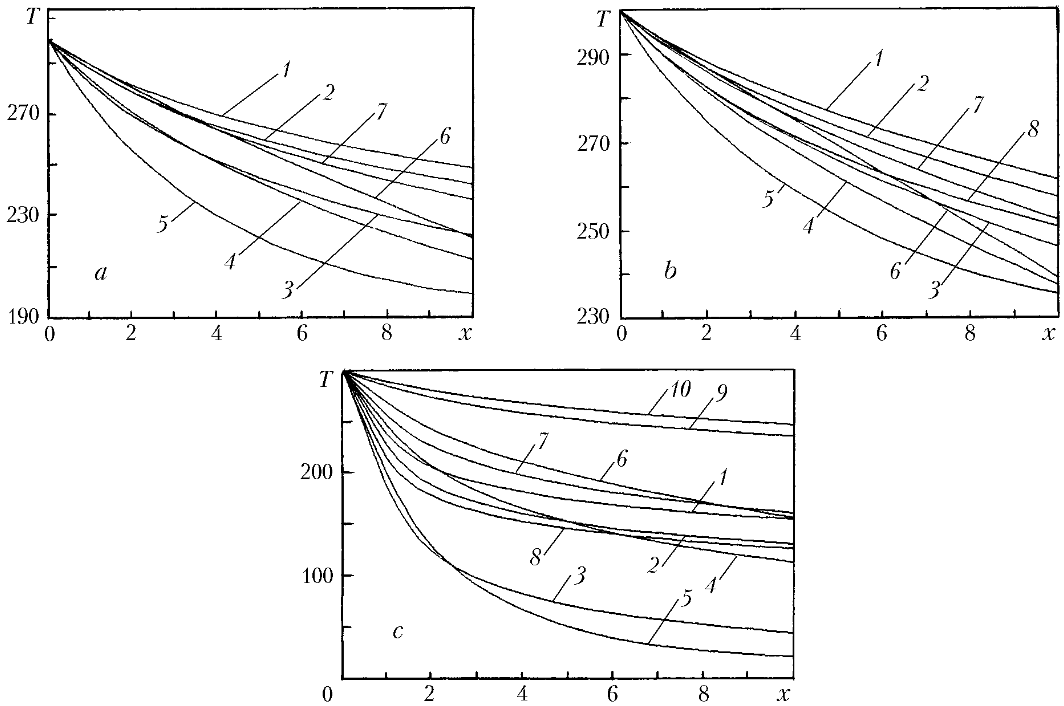

Fig. 1. Temperature-field distribution in a conic shell (a), an elliptic-paraboloid shell (b), and a one-sheeted-hyperboloid shell (c) at a definite temperature and a definite heat flow at one of the edges of the shell.

(meridional-peripheral reinforcement) to 278.7°C (fifth-type reinforcement) and the temperature difference at the edge x_1^1 comprises 138.8°C.

This significant influence of the type of reinforcement of a shell of revolution on the temperature distribution in it is apparently explained by the significant influence of the radius $R(x_1)$ on the distribution of the heat flow in the shell. Actually, it follows from (28) that the meridional component of the heat flow in the shells considered is determined by the formula $q_1(x_1) = q_1(x_1^0)R_0/R(x_1)$ and, if $R(x_1) < R_0$, then $q_1(x_1) > q_1(x_1^0)$ and, consequently, the temperature change at the point x_1, characterized by the derivative $T'(x_1)$, will be larger than the temperature change at the edge x_1^0 ($T'(x_1) > T'(x_1^0)$), all other things being equal (see (28)). If $R(x_1) > R_0$, then $q_1(x_1) < q_1(x_1^0)$ and the temperature change at the point x_1 will be smaller than the temperature change at the edge x_1^0 ($T'(x_1) < T'(x_1^0)$). Consequently, in the first case, the temperature of a shell changes rapidly by a large value and, in the second case, the temperature of a shell changes slowly by a small value. For example, in a one-sheeted-hyperboloid shell ($c = 1.5R_0$, see curves 1–8 in Fig. 1c), the radius $R(x_1)$ initially decreases from R_0 to $R(c)$ with increase in x_1 (the value of the parameter c in (25) determines the positions of points on the throat line of the middle surface of this shell [13]) and then increases to R_1; however, in this case, the values of the radius $R(x_1)$ do not exceed R_0 in the range $x_1^0 \le x_1 \le 2c - x_1^0 = 3R_0$. Therefore, the temperature of the shell considered changes significantly in the indicated range, which influences the behavior of curves 1–8 (Fig. 1c). At $c = 0$ and $c = -2R_0$ (curves 9 and 10 respectively), the throat line coincides with the edge x_1^0 ($c = 0$) or lies outside of the shell ($c = -2R_0$); therefore, in both cases, $R(x_1) > R_0$ at $x_1 > x_1^0$, which explains the more smooth behavior of curves 9 and 10, e.g., as compared to the behavior of curve 8, even though all three corresponding composites were reinforced along the asymptotic directions. Curve 9 lies somewhat lower than curve 10 because in the shell to which curve 9 corresponds, the radius $R(x_1)$ changes "slowly" in the neighborhood of the edge x_1^0 ($R'(x_1^0) = R'(c) = 0$) and in the shell to which curve 10 corresponds, the radius $R(x_1)$ changes "rapidly" in the neighborhood of this edge since $R'(x_1^0) > 0$. Therefore, the behavior of curves 9 and 10 differs only in the neighborhood of the edge x_1^0 and, at points located at large distances from the edge x_1^0, these lines are practically identical to equidistant lines.

The fact that the temperatures of conic shells (Fig. 1a) and elliptic-paraboloid shells (Fig. 1b) with reinforcement structures 1–7 differ not so significantly than the temperatures of one-sheeted-hyperboloid shells ($c = 1.5R_0$, Fig.

1c) with the same reinforcement structures is explained analogously. In both conic and elliptic-paraboloid shells, $R(x_1) > R_0$ at $x_1 > x_1^0$; therefore, $q_1(x_1) < q_1(x_1^0)$ and the temperatures change more smoothly in them than in one-sheeted hyperboloid shells in which $R(x_1) \le R_0$ at $x_1^0 \le x_1 \le 2c - x_1^0$ and, as a consequence, $q_1(x_1) \ge q_1(x_1^0)$. The more smooth behavior of the curves in Fig. 1b as compared to the curves in Fig. 1a is explained by the same facts. The radius R in a conic shell changes in proportion to x_1 ($R'(x_1)$ = const). In an elliptic-paraboloid shell ($c = -0.01R_0$), R increases "rapidly" and significantly in the neighborhood of the edge x_1^0 ($R'(x_1^0) \xrightarrow{c \to -0} +\infty$) and, in the regions far removed from this edge, R changes insignificantly, with the result that, even in the neighborhood of the point x_1^0, the heat flow q_1 decreases significantly, which explains the smaller temperature difference at the edges of this shell as compared to that in a conic shell.

It was already noted that an elliptic-paraboloid shell with $c \to -\infty$ and a one-sheeted-hyperboloid shell with $c \to c^0$ (see (23)–(26)) degenerate into conic shells; therefore, it is appropriate to follow the change in the temperature of the indicated shells at c values close to the limiting values at which these shells are close in shape to a conic shell. For example, curve 8 in Fig. 1b corresponds to a shell with $c = -10R_0$ with a meridional reinforcement structure. Comparison of this curve with curve 1 in Fig. 1a (conic shell with a meridional reinforcement structure) shows that the temperatures of the composites considered differ insignificantly and, at the edge x_1^1, the difference between them comprises only 2.7°C; curves 9 and 10 in Fig. 1c correspond to one-sheeted-hyperboloid shells with $c = 0$ and $c = -2R_0$, which are reinforced along asymptotic directions (at $c \to c^0 = -2.5R_0$, such a shell degenerates into a conic shell and the asymptotic reinforcement directions tend to meridional ones: $\alpha_k \to 0$). Comparison of the indicated curves with curve 1 in Fig. 1a indicates once again that the temperatures in the composites considered differ insignificantly; at the edge x_1^1, the difference between them comprises −13 and −1.4°C respectively. Consequently, when the parameter c in (23)–(26) tends to a limiting value, the temperatures in an elliptic-paraboloid shell and a one-sheeted-hyperboloid shell, having corresponding reinforcement structures, tend to the temperature of a conic shell in a downward and an upward direction, respectively.

The shape of a shell influences its temperature field not only quantitatively but also qualitatively. For example, for a conic shell (Fig. 1a) and an elliptic-paraboloid shell ($c = -0.01R_0$, Fig. 1b), to the maximum temperatures at the edge x_1^1 correspond curves 1, 2, and 7, and to the minimum temperatures at this edge correspond curves 6, 4, and 5 (in decreasing order); for a one-sheeted-hyperboloid shell ($c = 1.5R_0$, Fig. 1c), to the maximum temperatures at the indicated edge correspond curves 7, 6, and 1, and to the minimum temperatures at this edge correspond curves 4, 3, and 5. Consequently, if any reinforcement structure is best for any shell in certain thermophysical parameters, another reinforcement structure can be best for a shell with a different geometry.

If the requirement of a minimum temperature at the edge x_1^1 is used as a thermophysical criterion, as is seen from Fig. 1, the fifth-type reinforcement structure (peripheral reinforcement with a linear distribution of the density ω_k (32) and the conditions $\omega_k^0 \ne 0$ and $\omega_k^1 = 0$) will be best in this sense. If, in addition, the geometry of a shell is considered, a one-sheeted-hyperboloid shell with the fifth type of applying a winding will be best; in this case, $T(x_1^1)$ = 21.3°C, i.e., room temperature is practically attained.

If the requirement of a maximum total heat

$$Q = \int_V C\hat{T}dV = 2\pi H \int_{x_1^0}^{x_1^1} C(x_1)\, \hat{T}(x_1)\, H_1(x_1)\, R(x_1)\, dx_1 \tag{33}$$

(the function C is determined from (3)) accumulated by a reinforced shell is used as a criterion of its efficiency, an elliptic-paraboloid shell with a meridional reinforcement structure (curve 1, Fig. 1b) will be best. A one-sheeted-hyperboloid shell with a fifth-type reinforcement (curve 5, Fig. 1c) has the smallest value of Q determined from (33). Consequently, different criteria point to different-geometry shells with different reinforcement structures as most effective.

However, not only the reinforcement structure and geometry of a shell, but also the boundary heat conditions influence the temperature-field distribution in it. We will demonstrate this by the following example. Let us consider a one-sheeted-hyperboloid shell with the above-indicated geometric parameters and temperatures $T(x_1^0)$ = 300°C and $T(x_1^1)$ = 20°C at both of its edges. Figure 2 presents temperature distributions in such shells with different reinforcement structures (the enumeration of the curves corresponds to that in Fig. 1c). Comparison of the curves in Figs. 1c

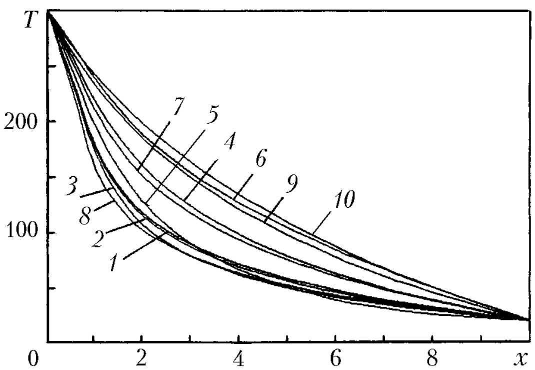

Fig. 2. Temperature-field distribution in a one-sheeted-hyperboloid shell at definite temperatures at both of its edges.

and 2 shows that a change in the boundary conditions leads to both qualitative and quantitative changes in the temperature field of a shell. For example, to the lowest temperatures in the neighborhood of the edge x_1^0 corresponds curve 3 in Fig. 1c and curve 8 in Fig. 2; curves 9 and 10 in Fig. 1c differ significantly from curve 6 in Fig. 2, and these curves in Fig. 2 differ insignificantly from this curve, and so on. Calculation, by (38), of the total amount of heat accumulated in one-sheeted-hyperboloid shells ($c = 1.5R_0$) with different reinforcement structures shows that, in the first example (Fig. 1c), a maximum value of Q is accumulated in a shell with a sixth-type reinforcement structure and a minimum value of Q is accumulated in a shell with a fifth-type reinforcement structure; in the second example, a maximum Q is also accumulated in a shell with a sixth-type reinforcement structure and a minimum Q is accumulated in a shell with a third-type reinforcement structure. Consequently, a reinforcement structure that is most effective at some boundary conditions can be not so effective at other boundary conditions.

Our investigation has shown that the temperature field of fibrous composites such as shells of revolution substantially depends qualitatively and quantitatively on their reinforcement structure (parameters α_k and ω_k), the thermophysical characteristics of the composite phases (λ_m and λ_k, $k = 1, 2$), the geometry of a shell ($R(x_1)$), and the boundary heat conditions, which opens up a wide range of ways of searching for effective designs of reinforced composites and their geometries. Problems on the search for the most effective, in thermophysical parameters, reinforcement structures should be formulated individually for different-geometry shells and for different heat actions. Evidently, the above-described features will also be characteristic of shells with a more complex geometry and in the case of more complex heat actions than those considered in the present work, which generates a need for developing efficient numerical and analytical methods for calculating actual reinforcement structures, taking into account their features.

In summary, it may be said that all the conclusions drawn in the present work are true for the case where the thermophysical characteristics of the phases of a composite are dependent on the temperature; in this case, the sole difference between the cases considered is that Eqs. (27) and (28) will be not linear but quasilinear and the heat capacity C in (33) will be dependent on x_1 and $\hat{T}$.

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NOTATION

c, parameter determining the geometry of a shell of revolution, m; C, reduced heat capacity of the composite material of a shell, J/K; c_m and c_k ($k = 1, 2, ..., N$), specific heats of the materials of the binding matrix and the kth reinforcements, J/(kg·K); H_i ($i = 1, 2, 3$), Lamé parameters in the curvilinear orthogonal coordinate system x_i, m; K, Gaussian curvature of a shell, 1/m^2; L, parabolic differential operator of the heat-conduction equation (1); N, number of reinforcing-fiber families; Q, total amount of heat accumulated in a shell, J; q_*, integration constant representing the product of the meridional component of the heat-flow vector into R, W/m; q_1, meridional component of the heat-flow vector, W/m^2; q_3, component of the heat-flow vector in the direction x_3, W/m^2; R, distance from a point of the middle surface of a shell to its rotation axis, m; R_0 and R_1, radii of a shell at its left $x_1 = x_1^0$ and right $x_1 = x_1^1$ edges, m; T, averaged temperature of the composite material of a shell, ^{o}C; $\hat{T} = T + 273$, absolute temperature of the composite material of a shell, K; t, time, sec; $T_{+\infty}$ and $T_{-\infty}$, ambient temperatures on the "outer" and "inner" faces of a shell, ^{o}C; W, reduced power density of the internal heat sources, W/kg; w_m and w_k (k = 1, 2 , ..., N), power densities of the internal heat sources in the binding matrix and in the kth reinforcements, W/kg; V, volume of a composite, m^3; $x = x_1/R_0$, independent variable along the rotation axis of a shell; x_i (i = 1, 2, 3), spatial curvilinear orthogonal coordinates of points on a shell; x_1^0 and x_1^1, values of the variable x_1 along the rotation axis of a shell which determine the positions of its left and right edges; y_1, y_2, y_3, Cartesian coordinate system, m; α_k (k = 1, 2, ..., N), angle between the kth reinforcement fiber and the direction x_1, rad; Λ_{ij} (i, j = 1, 2, 3), effective heat-conductivity coefficients of a shell, W/(m·K); λ_m and λ_k (k = 1, 2, ..., N), linear heat-conductivity coefficients of the isotropic materials of the binding matrix and the kth fibers, W/(m·K); μ_+ and μ_-, coefficients of heat exchange between a shell and the environment on the "outer" and "inner" faces of the shell, W/(m^2·K); ρ_m and ρ_k (k = 1, 2, ..., N), volume densities of the materials of the binding matrix and the kth reinforcements, kg/m^3; Ω, total density of a reinforcement; Ω_k (k = 1, 2, ..., N), total consumption of kth reinforcements, m^3; ω_k (k = 1, 2, ..., N), density of the reinforcement with kth fibers; ω_k^0, ω_k^1, density of a reinforcement at the edges $x_1 = x_1^0$ and $x_1 = x_1^1$; prime, ordinary differentiation with respect to the variable x_1; lower index after the radix point, partial differentiation with respect to the time t or with respect to the corresponding spatial variable x_i. Subscripts: m, matrix; c, s, functions representing cofactors of the cosine and sine respectively; $\pm$, values of a function on the "outside" (+) or "inside" (–) faces of a plate.

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